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THE MINIMIZATION OF THE TOTAL PRESSURE LOSS ACCOMPANYING THE BREAKDOWN OF A SUPERSONIC FLOW†

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The problem of minimizing the total pressure loss accompanying the breakdown of a supersonic flow to subsonic velocities through a system of successively ordered shock waves is considered. By changing to the corresponding problem of non-linear programming with non-linear constraints in the form of inequalities, a point which is suspected of being an extremum is determined and it is proved that it is the point of a strict local minimum. It is noted that, when the number of shock waves increases to infinity, the optimal shock wave system changes into an isoentropic wave. © 1999 Elsevier Science Ltd. All rights reserved.

This problem arose in the middle of the 1940s in connection with the problem of constructing efficient supersonic air intakes. One of the first people to do work in this field was Osvatich (see the bibliography in [1]), who considered systems consisting of n oblique and a normal $(n + 1)$th closing shock wave. As regards minimizing the total pressure loss, such systems were found to be more effective than a single normal shock wave at any intensities of the first n oblique shock waves. In this case, for a specified free-stream Mach number and a specified number n of oblique shock waves, optimal values of the intensities of the oblique shock waves exist for which the total pressure losses are a minimum.

The results of Osvatich's work, which was carried out in Germany in 1943 and was classified, turned up in the USA after the end of World War II. In 1947, Petrov and Ukhov proposed a numerical solution of the problem (see [2]) and, unlike Osvatich, they not only considered systems with a normal closing shock wave, but also systems consisting of n oblique shock waves and a closing sonic shock wave. As it turned out, the latter ensures a better restoration of the total pressure [2].

In the middle of the 1980s, Petrov proposed to one of us that an attempt should be made to obtain an analytic solution of the problem, subject to the condition that the type of closing shock wave should not be fixed in advance but obtained as one of the results of the solution.

In this paper, such a solution is now presented at the stage of a preliminary analysis of the problem. The solution obtained is compared with Osvatich's results.

1. FORMULATION OF THE PROBLEM

A plane supersonic flow of a perfect inviscid gas, which successively traverses a system S_{n+1} consisting of $n + 1$ shock waves, is considered. It has been shown (see [3], for example) that, for a fixed free-stream Mach number and a fixed adiabatic exponent $\gamma \in (1, 2]$, the ratio of any gas dynamic variables in the system and preceding it is expressed in terms of the intensities $J_k = p_k/p_{k-1}$ of the shock waves. In particular, the total pressure loss factor $K_{n+1}^{(p_0)}$, which is the ratio of the total pressure downstream of the system S_{n+1} to the total pressure of the unperturbed flow, is defined by the formula

$$K_{n+1}^{(p_0)} = \prod_{k=1}^{n+1}\left[J_k\left(\frac{J_k+\varepsilon}{J_k(1+\varepsilon J_k)}\right)^{\lambda}\right], \quad \lambda = \frac{1+\varepsilon}{2\varepsilon}, \quad \varepsilon = \frac{\gamma-1}{\gamma+1} \tag{1.1}$$

One of the most important problems which is frequently encountered in practice is the problem of the breakdown of a supersonic flow through the system S_{n+1} to subsonic velocities with minimum losses in the total pressure, that is, the problem

$$K_{n+1}^{(p_0)} \to \sup_{M_{n+1} \leq 1} \tag{1.2}$$

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where M_{n+1} is the Mach number downstream of the system S_{n+1}, which is related to the free-stream Mach number M as follows [3]:

$$1+\varepsilon(M_{n+1}^2-1)=(1+\varepsilon(M^2-1))\Pi_{n+1}, \Pi_{n+1}=\prod_{k=1}^{n+1}\frac{J_k+\varepsilon}{J_k(1+\varepsilon J_k)} \tag{1.3}$$

Starting from Osvatich's work, we take "a priori" a normal shock wave with intensity

$$J_{n+1}^1=(1+\varepsilon)M_n^2-\varepsilon \tag{1.4}$$

downstream of which the flow is always subsonic, as the closing $(n+1)$th shock wave and thereby ensure that condition $M_{n+1} \leq 1$ is satisfied. The J_k of the remaining shock waves are then found from the optimality conditions. However, such an approach does not lead to the optimal result (see [2], for example). Actually, for any number $M_n > 1$, a closing shock wave exists which ensures the breakdown of the flow to a velocity equal to the velocity of sound. The intensity of this shock wave is calculated from the formula [5]

$$J_{n+1}^0=\frac{\mu_n-1}{2\varepsilon}+\left[\left(\frac{\mu_n-1}{2\varepsilon}\right)^2+\mu_n\right]^{1/2}, \quad \mu_n=1+\varepsilon(M_n^2-1) \tag{1.5}$$

It can be verified that $J_{n+1}^0 < J_{n+1}^1$. Consequently, any shock wave with an intensity from the range $[J_{n+1}^0, J_{n+1}^1]$ can be taken as the closing shock wave.

The aim of this paper is to find the solution of problem (1.2) when the intensity J_{n+1} of the closing shock wave lies in the above-mentioned range.

2. FORMALIZATION OF THE PROBLEM

We put $\mu = 1+\varepsilon(M^2-1)$. According to (1.3), the inequality $M_{n+1} \leq 1$ can be written in the equivalent form

$$-\mu\Pi_{n+1}+1 \geq 0 \tag{2.1}$$

However, constraint (2.1) on the values of J_k is not unique. Actually, Zemplen's theorem [4] imposes the additional constraints

$$J_k-1 \geq 0, \quad k=1,\ldots,n+1 \tag{2.2}$$

Moreover, for system S_{n+1} to exist, it is necessary that the Mach number in each of the first n shock waves should be greater than or equal to unity. This condition has the form

$$\mu\Pi_k-1 \geq 0, \quad k=1,\ldots,n \tag{2.3}$$

Finally, the intensity J_{n+1} of the closing shock wave must not exceed the intensity of the normal shock wave (1.4), which leads to the inequality

$$-(1+\varepsilon J_{n+1})+(1+\varepsilon)\mu\Pi_n \geq 0 \tag{2.4}$$

Constraints (2.1)–(2.4) fully describe the set of designs Ω for the extremal problem in question. However, some of these constraints are superfluous.

Actually, inequality (2.1) denotes that the intensity of the $(n+1)$th closing shock wave must be greater than or equal to the "sonic" intensity J_{n+1}^0 (1.5). However, this intensity is always greater than or equal to unity so that the intensity J_{n+1} of the closing shock wave automatically satisfies Zemplen's theorem. Consequently, when account is taken of (2.1), condition (2.2) can be rewritten as

$$J_k-1 \geq 0, \quad k=1,\ldots,n \tag{2.5}$$

Next, when account is taken of (2.5), only the last (when $k=n$) of all the inequalities (2.3) is of interest. The remaining inequalities are automatically satisfied since

$$(J_i+\varepsilon)/[J_i(1+\varepsilon J_i)] \leq 1 \text{ when } J_i \geq 1$$

Hence, the set of designs Ω of the extremal problem can finally be described by the following system of inequalities

$$J_k - 1 \geq 0, \quad k = 1, \dots, n \tag{2.6}$$

$$\mu\Pi_n - 1 \geq 0, \ -\mu\Pi_{n+1} + 1 \geq 0, \ -(1+\varepsilon J_{n+1}) + (1+\varepsilon)\mu\Pi_n \geq 0$$

The parameters occurring in this system satisfy the constraints

$$\mu > 1, \quad \varepsilon \in (0, 1/3] \tag{2.7}$$

Instead of $K_{n+1}^{(p_0)}$, it is convenient to introduce its reciprocal $I(\mathbf{J}) = 1/K_{n+1}^{(p_0)}$ as the objective function, where $\mathbf{J} = (J_1, \dots, J_{n+1})$. According to (1.1)

$$I(\mathbf{J}) = \prod_{k=1}^{n+1} \left[\frac{1}{J_k} \left(J_k \frac{1+\varepsilon J_k}{J_k+\varepsilon} \right)^{\lambda} \right] \tag{2.8}$$

We arrive at the final form for the required extremal problem

$$I(\mathbf{J}) \to \inf_{\mathbf{J}\in\Omega} \tag{2.9}$$

3. PRELIMINARY ANALYSIS

The set of designs Ω of problem (2.9) is not empty: the vector $\mathbf{J} = (1, \dots, 1, J_{n+1}^1)$ satisfies the constraints of problem (2.9) for any values of μ and ε from the ranges (2.7).

We now consider the third constraint of (2.6). This can be rewritten in the form

$$a(\mathbf{J}) := 1/\Pi_{n+1} \geq \mu \tag{3.1}$$

In this case, according to (2.8)

$$I(\mathbf{J}) = a^{\lambda}(\mathbf{J}) \prod_{k=1}^{n+1} \frac{1}{J_k} \tag{3.2}$$

When inequality (3.1) becomes an equality, the first factor on the right-hand side of (3.2) takes the minimum value. We also note that the objective function $I(\mathbf{J})$ is symmetric. This provides grounds for supposing that the components of the solution $\mathbf{J}^*$ are equal to one another. It may therefore be assumed that the point $\mathbf{J}^* = (x, \dots, x)$, at which relation (3.1) is satisfied as an equality, will be the point which is suspected of being the extremum. It can be verified that the components x of the point $\mathbf{J}^*$ are calculated from the formula

$$x = \frac{\alpha - 1}{2\varepsilon} + \left[\left(\frac{\alpha-1}{2\varepsilon} \right)^2 + \alpha \right]^{1/2}, \quad \alpha = \mu^{1/(1+n)} \tag{3.3}$$

We will now show that, at the point $\mathbf{J}^*$, the first two and the last constraints are not active. As regards the first constraint, this is obvious since $x > 1$. The left-hand sides of the second and the last inequalities at the point $\mathbf{J}^*$ are respectively equal to

$$\frac{x(1+\varepsilon x)}{x+\varepsilon} - 1 \text{and} (1+\varepsilon x)\left[\frac{(1+\varepsilon)x}{x+\varepsilon} - 1 \right]$$

They are positive when $x > 1$. Hence, only the third constraint is active at the point $\mathbf{J}^*$.

4. STRICT LOCAL OPTIMALITY

We will now consider the auxiliary problem

$$I(\mathbf{J}) \to \inf, \quad a(\mathbf{J}) - \mu \geq 0 \tag{4.1}$$

and show that the point $\mathbf{J}^* = (x, \ldots, x)$, where x is calculated using formula (3.3), is the point of a strictly local minimum in the case of this problem. It will follow in an obvious way from here that $\mathbf{J}^*$ is the point of a strict local minimum in the case of problem (2.9) also.

We recall that the constraint $a(\mathbf{J}) - \mu \geq 0$ is active at the point $\mathbf{J}^*$. We make use of the sufficient conditions for a strict local minimum for a linear programming problem [6]. A strict local optimality of $\mathbf{J}^*$ will be established if we find a positive number u^* such that $I'(\mathbf{J}^*) = u^* a'(\mathbf{J}^*)$ and

$$\langle (I''(\mathbf{J}^*) - u^* a''(\mathbf{J}^*))\mathbf{g}, \mathbf{g}\rangle > 0 \tag{4.2}$$

for all non-zero vectors g from the tangential subspace

$$\langle a'(\mathbf{J}^*), \mathbf{g}\rangle = 0 \tag{4.3}$$

We will denote the partial derivative of the function $a(\mathbf{J})$ with respect to J_k by $a_k'(\mathbf{J})$. By (3.1), we have

$$\begin{gathered} a_k'(\mathbf{J}) = \varepsilon b(J_k)c(J_k)a(\mathbf{J}) \\ b(t) = t^2 + 2\varepsilon t + 1, \quad c(t) = [t(t+\varepsilon)(1+\varepsilon t)]^{-1} \end{gathered} \tag{4.4}$$

In particular

$$a_k'(\mathbf{J}^*) = \varepsilon b(x)c(x)a(\mathbf{J}^*) \tag{4.5}$$

We immediately note that condition (4.3) is equivalent to the following

$$\sum_{k=1}^{n+1} g_k = 0 \tag{4.6}$$

Next, by (3.2), (4.4) and the definition of λ

$$I_k'(\mathbf{J}) = [\lambda a^{\lambda-1}(\mathbf{J})a_k'(\mathbf{J}) - a^{\lambda}(\mathbf{J})J_k^{-1}]\prod_{i=1}^{n+1} J_i^{-1} = \frac{1-\varepsilon}{2}(J_k - 1)^2 c(J_k)I(\mathbf{J}) \tag{4.7}$$

In particular

$$I_k'(\mathbf{J}^*) = \frac{1-\varepsilon}{2}(x-1)^2 c(x)I(\mathbf{J}^*) \tag{4.8}$$

We put

$$u^* := \frac{1-\varepsilon}{2\varepsilon}\frac{(x-1)^2}{b(x)}\frac{I(\mathbf{J}^*)}{a(\mathbf{J}^*)} > 0 \tag{4.9}$$

Then, by virtue of (4.5) and (4.8), the equality $I'(\mathbf{J}^*) = u^* a'(\mathbf{J}^*)$ will be satisfied.

We will now verify condition (4.2). When $j \neq k$, by (4.4) and (4.7), we have

$$\begin{gathered} a_{kj}''(\mathbf{J}) = \varepsilon^2 b(J_k)c(J_k)b(J_j)c(J_j)a(\mathbf{J}) \\ I_{kj}''(\mathbf{J}) = \frac{1}{4}(1-\varepsilon)^2(J_k - 1)^2 c(J_k)(J_j - 1)^2 c(J_j)I(\mathbf{J}) \end{gathered}$$

When $j = k$

$$\begin{gathered} a_{kk}''(\mathbf{J}) = \varepsilon c^2(J_k)a(\mathbf{J})[\varepsilon b^2(J_k) + \eta(J_k)] \\ I_{kk}''(\mathbf{J}) = \frac{1}{2}(1-\varepsilon)c^2(J_k)I(\mathbf{J})\left[\frac{1}{2}(1-\varepsilon)(J_k-1)^4 + \rho(J_k)\right] \\ \eta(t) = 2(t+\varepsilon)c^{-1}(t) - b(t)[c^{-1}(t)]', \quad \rho(t) = 2(t-1)c^{-1}(t) - (t-1)^2[c^{-1}(t)]' \end{gathered}$$

In particular

$$a''(\mathbf{J}^*) = \varepsilon c^2(x)a(\mathbf{J}^*)[\varepsilon b^2(x)C + \eta(x)E]$$

$$I''(\mathbf{J}^*) = \frac{1}{2}(1-\varepsilon)c^2(x)I(\mathbf{J}^*)\left[\frac{1}{2}(1-\varepsilon)(x-1)^4 C + \rho(x)E\right]$$

where C is a matrix with elements which are all equal to unity and E is the identity matrix. On taking account of equalities (4.6) and (4.9), we obtain

$$\langle (I''(\mathbf{J}^*) - u^* a''(\mathbf{J}^*))\mathbf{g}, \mathbf{g}\rangle = \xi \|\mathbf{g}\|^2 \tag{4.10}$$
$$\xi = (1-\varepsilon^2)(x^2-1)b^{-1}(x)c(x)I(\mathbf{J}^*)$$

It is obvious that $\xi > 0$. Inequality (4.2) now follows from (4.10).

It has therefore been proved that sufficient conditions for optimality are satisfied at the point $\mathbf{J}^*$. We note that, in non-linear programming, results of this type can only rarely be obtained successfully.

5. ANALYSIS OF THE RESULT

It has been shown above that, when the intensity of the closing shock wave belongs to the range $[\mathbf{J}_{n+1}^0, \mathbf{J}_{n+1}^1]$, the solution $\mathbf{J}^*$ of the problem has the form

$$J_k^* = \frac{\alpha-1}{2\varepsilon} + \left[\left(\frac{\alpha-1}{2\varepsilon}\right)^2 + \alpha\right]^{1/2}, \quad k = 1, \ldots, n+1 \tag{5.1}$$

In this case, the flow velocity downstream of the last closing shock wave is exactly equal to the velocity of sound and $\mathbf{J}_{n+1} = \mathbf{J}_{n+1}^0$.

Osvatich considered systems in which the closing shock wave is normal, that is, systems with $\mathbf{J}_{n+1} = \mathbf{J}_{n+1}^1$. For such systems, a minimum loss in the total pressure is achieved, if the intensities of the first n oblique shockwaves are equal to one another ($J_1 = \ldots = J_n = J_*$) and they are determined from the equation

$$\mu = A\frac{J_*^{n-1}(1+\varepsilon J_*)^n}{(J_*+\varepsilon)^n}[J_*^2 + 2BJ_* + 1 + (J_* - 1)(J_*^2 + 2CJ_* + 1)^{1/2}]$$
$$A = \frac{\varepsilon(2+\varepsilon)}{4(1+\varepsilon)^2}; \quad B = \frac{\varepsilon^2 + 2\varepsilon + 2}{\varepsilon(2+\varepsilon)}; \quad C = \frac{\varepsilon(3\varepsilon+4)}{(2+\varepsilon)^2} \tag{5.2}$$

We now see how much the total pressure loss factor $K_{n+1}^{(p_0)}$ in a system with intensities (5.1) differs from the loss factor in the optimal system (5.2) with a normal closing shock wave in the case when $M = 3$, $\gamma = 1, 4$ and $n = 1$. With these values of the parameters, the intensities of all the shock waves in the system, calculated using formula (5.1), are equal to 3.591. In the optimal system with the normal closing shock wave, the intensity of the first shock wave is equal to 4.341 and the intensity of the normal closing shock wave is equal to 3.841. Here, the total pressure loss factor is equal to 0.664 in the first case and 0.581 in the second case.

6. THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION

We return to formula (3.3). The magnitudes of α and x depend on $n+1$, and we shall therefore employ the notation α_{n+1} and x_{n+1}. It is obvious that

$$\lim_{n\to\infty} x_n = 1, \quad \lim_{n\to\infty} n(\alpha_n - 1) = \ln\mu \tag{6.1}$$

We will now show that

$$\lim_{n\to\infty} (x_n)^n = \mu^\lambda \tag{6.2}$$

The quantity x_n satisfies the relation

$$x_n(1+\varepsilon x_n) = \alpha_n(x_n + \varepsilon)$$

which, after some reduction, can be rewritten in the form

$$x_n = 1 + (\alpha_n - 1)\upsilon_n, \quad \upsilon_n = \frac{1}{1+\sqrt{\alpha_n}} + \frac{x_n}{\varepsilon(x_n + \sqrt{\alpha_n})} \tag{6.3}$$

According to the first relation of (6.1), we have

$$\lim_{n\to\infty} \upsilon_n = \frac{1+\varepsilon}{2\varepsilon} = \lambda \tag{6.4}$$

It remains to raise both sides of equality (6.3) to the power n, take the limit as $n \to \infty$ and use the second relation of (6.2) and (6.4).

We will now present a physical interpretation of the results. The ratio J_σ of the static pressures downstream and upstream of the system S_n of n shock waves, which is sometimes called the intensity of the system [3], is expressed in terms of the intensities J_k of the shock waves using the formula

$$J_\sigma = \frac{p_n}{p} = \prod_{k=1}^{n} \frac{p_k}{p_{k-1}} = \prod_{k=1}^{n} J_k$$

Consequently, the quantity $(x_n)^n$, which, by definition, is equal to the product of the intensities of the shock waves in the optimal system $S_n^{(p_0)}$, is also the ratio of the pressures downstream and upstream of the system $S_n^{(p_0)}$.

It is well known that the breakdown of a supersonic flow to a velocity equal to the velocity of sound can also be brought about through a simple compression wave w [4]. The intensity of such a wave is related to the free-stream Mach number as follows:

$$J_w = \mu^\lambda \tag{6.5}$$

In this case, unlike in the case of shock waves, the total pressure loss in it generally does not exceed $J \equiv 1$, that is, it is optimal from this point of view and ideally solves problem (1.2).

Formulae (6.2) and (6.5) therefore show that, as the number of shock waves in the optimal system $S_n^{(p_0)}$ increases up to infinity, the pressure drop in the optimal system does not simply tend to a finite value but it tends to the intensity of the optimal isentropic wave J_w. Moreover, the pressure drop in an individual shock wave tends to unity (see the first relation of (6.1)) so that the shock wave degenerates into a weak discontinuity [4]. Consequently, a qualitative transition occurs when $n \to \infty$ and the system $S_n^{(p_0)}$ is converted into the optimal isentropic wave.

The arguments which have been presented together with the results in [3] enable us to treat any system consisting of n shock waves as a certain rough model of an isentropic wave. For fixed n, this model is most accurate when the intensities of the shock waves occurring in the system are equal to one another.

The above arguments have been presented previously at an intuitive level. In this paper, they have received a rigorous justification.

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